Effect of Head-Tail Mode Diffusion due to Synchrotron Radiation on Transverse Coupled-Bunch Instabilities with Harmonic-Cavity-Lengthened Bunches

F. J. Cullinan* and P. F. Tavares

MAX IV Laboratory, Fotongatan 2
Lund, 225 94, Sweden.
**E-mail: francis.cullinan@maxiv.lu.se*
www.maxiv.lu.se

R. Nagaoka

Synchrotron SOLEIL, L'Orme des Merisiers,
Saint Aubin BP 48, 91192 Gif-sur-Yvette Cedex, France.

G. Skripka

CERN, Geneva 23,
CH-1211, Switzerland.

Many present and future storage-ring-based light sources make use of cavities at a harmonic of the main RF to lengthen the electron bunches. This reduces Touschek and intrabeam scattering to improve the beam lifetime and transverse emittances respectively [1]. These harmonic cavities are also beneficial in increasing the threshold currents of many collective instabilities in both longitudinal [2] and transverse planes. The latter case has previously been explored in detail making extensive use of Laclare's eigenvalue method [3]. Such an approach does not take into account the diffusion of head-tail modes of nonzeroth order, which is a consequence of the quantum-mechanical nature of synchrotron radiation [4,5]. In this paper, we show that Laclare's eigenvalue method is a change of basis from another popular method of solving the Sacherer formalism [4–6], where Fokker-Planck diffusion terms can be readily included [4,5]. For the case of the MAX IV 3 GeV ring [7], it is found that the effect is a large increase in the threshold currents of the resistive-wall coupled-bunch instability at positive chromaticity.

Keywords: Instabilities; Collective effects; Impedance; harmonic cavities; resistive-wall; synchrotron; diffraction-limited light-source.

1. Introduction

MAX IV is a light-source facility in Lund, Sweden [7]. It includes a 3 GeV electron storage ring, which is the first in a new generation of diffraction-limited light sources. The parameters of the 3 GeV ring are listed in Table 1. The multibend-achromat lattice requires strong-focusing small-bore magnets so the vacuum chamber must be small to fit within the magnet poles. This leads to a high resistive-wall impedance, which is expected to be the dominant impedance at most future ring-based light sources[a]. Harmonic cavities are installed in the MAX IV 3 GeV ring and they are

[a]It is also strong at MAX IV but higher-order-mode instabilities have been found to be dominant [8].

Table 1. The main parameters of the MAX IV 3 GeV storage ring in the MAX IV facility as used in this paper. The values correspond to the bare machine with no insertion devices.

Parameter	Value
Length L_c (m)	528.0
Design beam current (mA)	500
Radio frequency (MHz)	99.931
RF voltage (MV)	1.02
Passive cavity harmonic	3
Horizontal emittance (nm $\cdot$ rad)	0.3
Average vertical beta β (m)	6.95
Vertical betatron tune	16.28
Harmonic number h	176
Momentum compacion α_c	3.07×10^{-4}
Natural bunch length σ_τ (ps)	40.0
Lengthened bunch σ_τ (ps)	195
Energy spread σ_δ	7.69×10^{-4}
Energy loss per turn (keV)	363.8
Vertical chamber aperture $2a$ (mm)	22.0
Chamber resistivity ρ (Ωm)	1.7×10^{-8}

designed to be tuned so that their fields cancel out both the first and second derivatives of the main RF potential, providing the so-called flat-potential condition [9]. The effect that this has on transverse multibunch instabilities has been investigated thoroughly under the assumption that the growth rate of a coupled-bunch mode is proportional to the current and the damping rate is the same for all head-tail modes [3]. In reality, quantum emission of synchrotron radiation and subsequent quantum excitations lead to the diffusion of head-tail modes excited in the electron bunch and the rate of the diffusion is dependent on the head-tail mode order [4,5]. The different azimuthal head-tail modes can also couple when they are detuned close to each other in frequency and this also breaks the proportionality [6] but this is expected to be a small effect since the current of each bunch in a uniformly-filled storage ring is too small for significant detuning.

Laclare's eigenvalue method can be used to calculate the growth-rates of coupled-bunch instabilities under the assumption of simple-harmonic motion in longitudinal phase space and no coupling between the different head-tail modes [10]. It is an eigenvalue problem where the eigenvector is the bunch spectrum and the eigenvalue is the complex coherent betatron tune shift, where the imaginary part of the tune shift is the growth rate. The results obtained using Laclare's eigenvalue method have been shown to agree very well with macroparticle tracking simulations [3]. It does not include the effects of head-tail-mode diffusion or mode-coupling but, as outlined in the next section, it can be shown that it is merely a change of basis from another common frequency-domain method that has been developed to include such effects. The derivation follows on from the outline of Laclare's eigenvalue method in the previous study [3]. More complete derivations of Laclare's eigenvalue method can be found elsewhere[10,11].

2. Change of Basis

A matrix can be transformed to a different basis by performing a change of basis operation

$$\mathbf{B} = \mathbf{S}\mathbf{A}\mathbf{S}^{-1} \tag{1}$$

where **A** and **B** are the same matrix in two different bases and **S** is the change of basis matrix from the basis of **A** to the basis of **B**. A matrix has the same eigenvalues in all bases and its eigenvectors can be transformed from one basis to another by multiplying by the change of basis matrix.

The bunch spectrum, the eigenvector of Laclare's eigenvalue method σ_m for azimuthal head-tail mode number m, is defined as

$$\sigma_m(\omega_{mp}) = 2\pi \int_0^\infty J_m[(\omega_{mp} - \omega_\xi)r] g_0(r) Y_m(r) r dr \tag{2}$$

where ω_ξ is the chromatic frequency, r is the amplitude of synchrotron oscillation in units of time, g_0 is the bunch charge distribution, $Y(r)$ is the amplitude of betatron oscillation and J_m is the Bessel function of order m. ω_{mp} is an infinite series of the discrete angular frequencies that the beam excites given by $\omega_{mp} = (Mp+\mu)\omega_0 + \omega_\beta + m\omega_s$ where p is an integer, M is the number of bunches, μ is the coupled-bunch mode number, ω_0 is the angular revolution frequency and ω_β and ω_s are the synchrotron and betatron angular frequencies respectively[3]. Equation 2 can be expressed as a sum of Laguerre polynomials $L_k^{(m)}$ using the following relation:

$$J_m(x) = \left(\frac{x}{2}\right)^m \frac{e^{-t}}{m!} \sum_{k=0}^{\infty} \frac{L_k^{(m)}\left(\frac{x^2}{4t}\right)}{L_k^{(m)}(0)} \frac{t^k}{k!} \,. \tag{3}$$

If the arbitrary parameter t is chosen to be

$$t(p) = (\omega_{mp} - \omega_\xi)^2 \frac{\sigma_\tau^2}{2} \,, \tag{4}$$

where σ_τ is the bunch length in time, a change of basis matrix can be defined, whose elements are given by

$$S_{pk}^{-1} = \frac{2^{\frac{|m|}{2}} t(p)^{\frac{|m|}{2}+k} |m|! e^{-t(p)}}{\sqrt{k!(k+|m|)!}} \tag{5}$$

where k is the radial head-tail mode number. The eigenvector in the new basis is then

$$\sigma_m(k) = 2\pi \int_0^\infty \left[\left(\frac{r}{\sqrt{2}\sigma_\tau}\right)^{|m|} \sqrt{\frac{k!}{(k+|m|)!}} L_k^{(|m|)}\left(\frac{r^2}{2\sigma_\tau^2}\right) \right] g_0(r) Y_m(r) r dr \,. \tag{6}$$

The frequency basis of Laclare has been transformed to a basis of radial head-tail modes described by Laguerre polynomials $L_k^{(|m|)}$. The Physics, however, remains the same. The new basis is the same one used by Chin[6] and Suzuki[4]. The previous analysis[3] can therefore be extended freely to include the effects of head-tail mode coupling and diffusion by switching to the new basis and adding the relevant terms. The diffusion of head-tail modes is added as an additional damping term such that the total damping rate $1/\tau_{m,k}$ for the head-tail mode of azimuthal order m and radial order k is given by

$$\frac{1}{\tau_{m,k}} = \frac{1}{\tau_y} + \frac{2k+m}{\tau_0} \tag{7}$$

where τ_y and τ_0 are the damping times due to synchrotron radiation in the vertical and longitudinal planes respectively. Unlike Laclare's eigenvalue method, the new basis has only been developed to treat bi-Gaussian bunch-charge distributions in synchrotron phase space. The inverse of the change of basis matrix can also be used as a simple way to obtain the bunch spectrum by converting the eigenvector with the largest growth rate from the new basis back to the frequency basis. This can be useful for identifying the dominant features of the machine impedance.

3. Results

The effect of the diffusion of head-tail modes due to quantum emission and excitation was investigated using macroparticle simulations of the flat-potential condition in the tracking code Mbtrack[12]. As for the previous study[3], the simulations are for the vertical plane in the MAX IV 3 GeV ring and only the resistive-wall impedance is included. Two cases were simulated. In the first case, no radiation damping is included and the growth-rate of the resistive-wall instability is measured. The threshold current is then taken as the current at which the growth rate would be equal to the radiation damping, assuming that the growth rate is directly proportional to the current. In the second case, the effects of radiation damping and quantum excitation were included in the simulation. The results are shown in Fig. 1. At high chromaticity, the increase in the threshold current with chromaticity is linear in both cases but the slope is almost a factor of four higher in the case where the head-tail mode diffusion is included. Although there is some deviation from the linear trend at a chromaticity of around 0.5, the peak seen without the diffusion included is no longer present with the consequence that at this chromaticity, there is even a decrease in the threshold current. At zero chromaticity, inclusion of the diffusion has no effect.

Fig. 2 shows the results of frequency domain calculations where the basis of Laguerre polynomials has been used and the effects of radiation damping and diffusion have been included. A comparison is made to the results of macroparticle tracking in the time domain. The results are for Gaussian bunches of two different lengths: the natural bunch length for MAX IV and the bunch length expected in the flat potential. The lengthening in the time-domain simulations has been achieved by lowering the RF voltage and neglecting the energy loss to synchrotron radiation

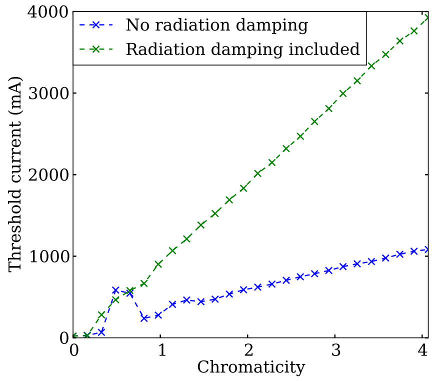

Fig. 1. Threshold currents of the resistive-wall instability with a harmonic-cavity-flattened potential as determined using macroparticle simulations with and without the effects of head-tail mode diffusion included.

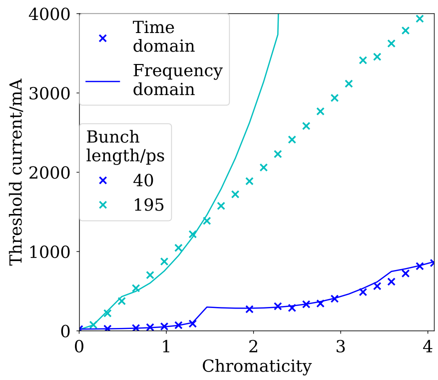

Fig. 2. Threshold currents of the resistive-wall instability as determined using macroparticle simulations in the time domain and the Laguerre-polynomial basis in the frequency domain with the effects of head-tail mode diffusion included.

(while keeping the same damping and randomised quantum effects). Comparison of Fig. 1 and Fig. 2 suggests that the difference between the results with Gaussian bunches lengthened in this way and an accurate simulation of the harmonic-cavity-flattened potential is very small. The transitions between the different azimuthal head-tail modes, which show up as discontinuities in the frequency-domain curve for the 40 ps bunch length in Fig. 2, are clear for the natural bunch length but are less clear for the lengthened bunches. This is because, with long bunches and high chromaticity, the radial order becomes much larger than the azimuthal order and the different azimuthal head-tail modes have approximately the same threshold currents, as observed previously[3]. The agreement with the tracking results is excellent for the natural bunch length but is worse for the lengthened bunch at high chromaticity. This is because it becomes hard to include enough radial modes in the frequency domain calculations due to the numerical difficulty of calculating large factorials.

Fig. 3 shows the bunch spectrum extracted from time domain simulations using

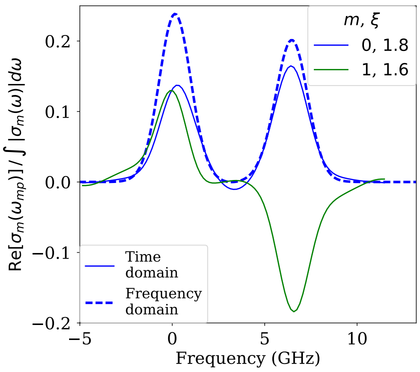

Fig. 3. Bunch spectra as determined from macroparticle simulations in the time domain and by multiplying the eigenvector from the frequency domain calculation with the highest growth rate by the change of basis matrix.

modal analysis[3] at two different chromaticities where different azimuthal head-tail modes are dominant. At the chromaticity of 1.8, the bunch spectrum has also been obtained from the frequency domain results by multiplying the eigenvector with the fastest growth rate by the inverse change of basis matrix $\mathbf{S}^{-1}$. As previously observed[3], at both chromaticities, the head-tail mode that dominates has the radial order that maximises the amplitude of the bunch spectrum at zero frequency where the amplitude of the resistive-wall impedance is largest. Introducing the head-tail mode diffusion has not affected this conclusion.

Fig. 4 shows the bunch spectra for bunches in a harmonic-cavity-flattened potential with and without head-tail mode diffusion, as simulated using macroparticle tracking. Once again, the bunch spectra have a common maximum amplitude close to zero frequency. However, in the case where the diffusion is included, the two largest peaks are slightly closer together indicating that the head-tail mode is of slightly lower order. This is a consequence of the higher-order head-tail modes being damped faster due to the diffusion. The bunch spectrum also has more features at frequencies above and below the frequencies of these peaks. The faster damping shows up in the frequency domain as a broader spectrum of radial head-tail modes. This broader spectrum of radial head-tail modes is also an explanation for the absence of the peak at a chromaticity of around 0.5 that shows up in the simulations without head-tail mode diffusion. More head-tail modes are excited so the transi-

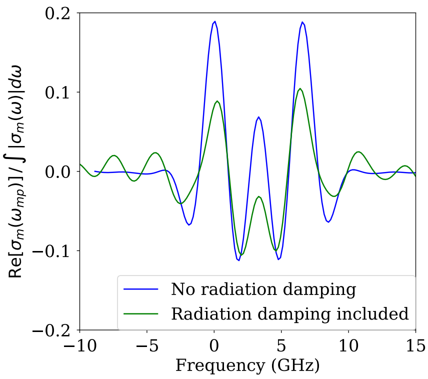

Fig. 4. Bunch spectra from modal analysis of the results of macroparticle simulations of the resistive-wall instability with a harmonic-cavity-flattened potential.

tion from a single-peaked bunch spectrum to a double-peaked bunch spectrum is much less pronounced.

4. Conclusion

Previously, four features of a bunch in the harmonic-cavity-flattened potential have been identified as reasons for its influence on the threshold currents of simulated resistive-wall coupled-bunch instabilities [3]. When the effects of diffusion of nonzeroth-order head-tail modes due to quantum emission and excitation is taken into account, only one of these features, the longer bunch length, appears to make a significant difference. Since the same threshold currents are found with a lengthened Gaussian bunch as with an accurate simulation of the dynamics in the harmonic-cavity-flattened potential, the other features: the synchrotron tune-spread, non-Gaussian bunch profile and nonradial bunch distribution in synchrotron phase space, are not significant. In the case of the MAX IV 3 GeV ring, the effect of head-tail mode diffusion is a large increase in the threshold current for all bunch lengths at positive chromaticity. As for the case with no diffusion, for long bunch lengths and high chromaticity, the threshold current increases linearly with the chromaticity and the limiting head-tail modes have much higher radial order than azimuthal. The bunch spectrum has a large peak centred at the frequency where the amplitude of the dominant impedance, in this case, the resistive-wall impedance, is largest.

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